Lecture 2

Propositional Logic & SAT

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Announcements

- Subscribe to the course mailing list at https://sympa.eng.utah.edu/sympa/info/sv
- Textbook

Syntax of Propositional Logic (PL)

truth_symbol ::= $\top$ (true), $\bot$ (false)

variable ::= p, q, r,…

atom ::= truth_symbol | variable

literal ::= atom | $\neg$atom

formula ::= literal |
$\neg$formula |
formula $\wedge$ formula |
formula $\vee$ formula |
formula $\rightarrow$ formula |
formula $\leftrightarrow$ formula

Examples of PL Formulae

F : $\top$

F : p

F : $\neg p$

F : $(p \wedge q) \rightarrow (p \vee \neg q)$

F : $(p \vee \neg q \vee r) \wedge (q \vee \neg r)$

F : $(\neg p \vee q) \leftrightarrow (p \rightarrow q)$

F : $p \leftrightarrow (q \rightarrow r)$

Semantics

- Semantics provides *meaning* to a formula
  - Defines mechanism for evaluating a formula
  - Formula evaluates to truth values *true*/1 and *false*/0
- Formula *F* evaluated in two steps
  1) Interpretation *I* assigns truth values to propositional variables
     $I : \{p \mapsto false, q \mapsto true \ldots\}$
  2) Compute truth value of *F* based on *I* using e.g. truth table
- formula *F* + interpretation *I* = truth value

Notation

- Let *F* be a formula and *I* an interpretation…
- *I* [*F*] denotes evaluation of *F* under *I*
- If *I* [*F*] = *true* then we say that
  - *F* is true in *I*
  - *I* satisfies *F*
  - *I* is a model of *F*

  and write $I \models F$
- If *I* [*F*] = *false* we write $I \not\models F$

Example

$F : (p \wedge q) \rightarrow (p \vee \neg q)$
$I : \{p \mapsto 1, q \mapsto 0\}$
(i.e., $I[p] = 1$, $I[q] = 0$)

p	q	$\neg q$	$p \wedge q$	$p \vee \neg q$	F
1	0	1	0	1	1

F evaluates to *true* under I or $I[F] = true$ or $I \models F$…

Satisfiability and Validity

- F is satisfiable iff (if and only if) there exists I such that $I \models F$
  - Otherwise, F is unsatisfiable
- F is valid iff for all I, $I \models F$
  - Otherwise, F is invalid
- We write $\models F$ if F is valid
- Duality between satisfiablity and validity:

  F is valid iff $\neg F$ is unsatisfiable

  Note: only holds if logic is closed under negation

Equivalence

- Two formulae F_1 and F_2 are equivalent, denoted by $F_1 \Leftrightarrow F_2$, iff they have the same models

Decision Procedure for Satisfiability

- Algorithm that in some finite amount of computation decides if given PL formula *F* is satisfiable
  - NP-complete problem
- Modern decision procedures for PL formulae are called *SAT solvers*
- Naïve approach
  - Enumerate truth table
- Modern SAT solvers
  - DPLL algorithm
    - Davis-Putnam-Logemann-Loveland
  - Operates on Conjunctive Normal Form (CNF)

Normal Forms

- Negation Normal Form (NNF)
  - Only allows ¬, ∧, ∨
  - Negation only in literals
- Disjunctive Normal Form (DNF)
  - Disjunction of conjunction of literals:

$$\bigvee_i \bigwedge_j l_{i,j}$$

- Conjunctive Normal Form (CNF)
  - Conjunction of disjunction of literals:

$$\bigwedge_i \bigvee_j l_{i,j}$$

Negation Normal Form

To transform F into F' in NNF recursively apply the following equivalences:

$$\neg\neg F_1 \Leftrightarrow F_1$$
$$\neg\top \Leftrightarrow \bot$$
$$\neg\bot \Leftrightarrow \top$$
$$\neg(F_1 \wedge F_2) \Leftrightarrow \neg F_1 \vee \neg F_2$$
$$\neg(F_1 \vee F_2) \Leftrightarrow \neg F_1 \wedge \neg F_2$$
$$F_1 \rightarrow F_2 \Leftrightarrow \neg F_1 \vee F_2$$
$$F_1 \leftrightarrow F_2 \Leftrightarrow (F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1)$$

Example

$F : p \leftrightarrow (q \rightarrow r)$

Conjunctive Normal Form

To transform F into F' in CNF first transform F into NNF and then recursively apply the following equivalences:

$$(F_1 \wedge F_2) \vee F_3 \Leftrightarrow (F_1 \vee F_3) \wedge (F_2 \vee F_3)$$
$$F_1 \vee (F_2 \wedge F_3) \Leftrightarrow (F_1 \vee F_2) \wedge (F_1 \vee F_3)$$

(Note: a disjunction of literals is called a clause.)

Example

$F : p \leftrightarrow (q \rightarrow r)$

Exponential Blow-Up

- Such a naïve transformation can blow-up exponentially (in formula size) for some formulae
  - For example: transforming from DNF into CNF

Tseitin Transformation [1968]

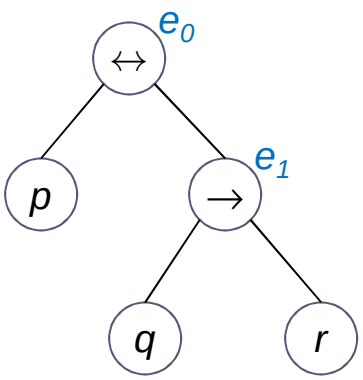
- Used in practice
  - No exponential blow-up
  - CNF formula size is linear wrt original formula
- Does not produce an equivalent CNF
- However, given *F*, the following holds for the computed CNF *F'*:
  - *F'* is equisatisfiable to *F*
  - Every model of *F'* can be translated (i.e., projected) to a model of *F*
  - Every model of *F* can be translated (i.e., completed) to a model of *F'*
- No model is lost or added in the conversion

Tseitin Transformation – Main Idea

- Introduce a fresh variable e_i for every subformula G_i of *F*
  - e_i represents the truth value of G_i
- Assert that every e_i and G_i pair are equivalent
  - Assertions expressed as CNF
- Conjoin all such assertions in the end

Example

$F : p \leftrightarrow (q \rightarrow r)$



Classical DPLL

- Searching for a model *M* for a given CNF formula *F*
  - Incrementally try to build a model *M*
  - Maintain state during search
- State is a pair *M* | *F*
  - *F* is a set of clauses and it doesn't change during search
  - *M* is a sequence of literals
    - No literals appear twice and no contradiction
    - Order does matter
    - Decision literals marked with l^d

Abstract Transition System

- Contains a set of rules of the form

$$M \mid F \Rightarrow M' \mid F'$$

denoting that search can move from state $M \mid F$ to state $M' \mid F'$

DPLL Rules – Extending *M*

- Propagate

$M \mid G, C \vee l \Rightarrow M, l \mid G, C \vee l$
if $M \models \neg C$ and l not in M

- Decide

$M \mid F \Rightarrow M, l^d \mid F$
if l or $\neg l$ in F and l not in M

DPLL Rules – Adjusting *M*

- Fail

$M \mid G,C \Rightarrow fail$
if $M \models \neg C$ and M contains no decision literals

- Backtrack

$M,l^d,N \mid G,C \Rightarrow M,\neg l \mid G,C$
if $M,l^d,N \models \neg C$ and N contains no decision literals

- Propagate

$M \mid G,C \vee l \Rightarrow M,l \mid G,C \vee l$
if $M \models \neg C$ and l not in M

- Decide

$M \mid F \Rightarrow M,l^d \mid F$
if l or $\neg l$ in F and l not in M

- Fail

$M \mid G,C \Rightarrow fail$
if $M \models \neg C$ and M contains no decision literals

- Backtrack

$M,l^d,N \mid G,C \Rightarrow M,\neg l \mid G,C$
if $M,l^d,N \models \neg C$ and N contains no decision literals

DPLL Example 1

$\emptyset$ | $\neg p \vee q \vee r$, p, $\neg q \vee r$, $\neg q \vee \neg r$, $q \vee r$, $q \vee \neg r$

DPLL Example 2

$\emptyset$ | $\neg p \vee q$, $\neg r \vee s$, $\neg t \vee \neg u$, $u \vee \neg t \vee \neg q$

DPLL Example 2

$\emptyset$ | $\neg p \vee q$, $\neg r \vee s$, $\neg t \vee \neg u$, $u \vee \neg t \vee \neg q$ $\Rightarrow$ (Decide p)

DPLL Example 2

$\emptyset$ | $\neg p \vee q$, $\neg r \vee s$, $\neg t \vee \neg u$, $u \vee \neg t \vee \neg q$ $\Rightarrow$ (Decide p)

p^d | $\neg p \vee q$, $\neg r \vee s$, $\neg t \vee \neg u$, $u \vee \neg t \vee \neg q$

DPLL Example 2

$\emptyset$ | $\neg p \vee q$, $\neg r \vee s$, $\neg t \vee \neg u$, $u \vee \neg t \vee \neg q$ $\Rightarrow$ (Decide p)

p^d | $\neg p \vee q$, $\neg r \vee s$, $\neg t \vee \neg u$, $u \vee \neg t \vee \neg q$ $\Rightarrow$ (Propagate q)

DPLL Example 2

$\emptyset$ | $\neg p \vee q$, $\neg r \vee s$, $\neg t \vee \neg u$, $u \vee \neg t \vee \neg q$ $\Rightarrow$ (Decide p)

p^d | $\neg p \vee q$, $\neg r \vee s$, $\neg t \vee \neg u$, $u \vee \neg t \vee \neg q$ $\Rightarrow$ (Propagate q)

p^d, q | $\neg p \vee q$, $\neg r \vee s$, $\neg t \vee \neg u$, $u \vee \neg t \vee \neg q$

DPLL Example 2

$\emptyset$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Decide p)
p^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Propagate q)
p^d, q	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Decide r)

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$\emptyset$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Decide p)
p^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Propagate q)
p^d, q	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Decide r)
p^d, q, r^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$

DPLL Example 2

$\emptyset$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Decide p)
p^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Propagate q)
p^d, q	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Decide r)
p^d, q, r^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Propagate s)

DPLL Example 2

$\emptyset$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Decide p)
p^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Propagate q)
p^d, q	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Decide r)
p^d, q, r^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q \Rightarrow$ (Propagate s)
p^d, q, r^d, s	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$

DPLL Example 2

$\emptyset$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide p)
p^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate q)
p^d, q	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide r)
p^d, q, r^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate s)
p^d, q, r^d, s	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide t)

DPLL Example 2

$\emptyset$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide p)
p^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate q)
p^d, q	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide r)
p^d, q, r^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate s)
p^d, q, r^d, s	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide t)
p^d, q, r^d, s, t^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	

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$\emptyset$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide p)
p^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate q)
p^d, q	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide r)
p^d, q, r^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate s)
p^d, q, r^d, s	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide t)
p^d, q, r^d, s, t^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate $\neg u$)

DPLL Example 2

$\emptyset$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide p)
p^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate q)
p^d, q	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide r)
p^d, q, r^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate s)
p^d, q, r^d, s	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide t)
p^d, q, r^d, s, t^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate $\neg u$)
$p^d, q, r^d, s, t^d, \neg u$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	

DPLL Example 2

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p^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate q)
p^d, q	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide r)
p^d, q, r^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate s)
p^d, q, r^d, s	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide t)
p^d, q, r^d, s, t^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate $\neg u$)
$p^d, q, r^d, s, t^d, \neg u$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Backtrack)

DPLL Example 2

$\emptyset$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide p)
p^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate q)
p^d, q	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide r)
p^d, q, r^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate s)
p^d, q, r^d, s	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide t)
p^d, q, r^d, s, t^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate $\neg u$)
$p^d, q, r^d, s, t^d, \neg u$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Backtrack)
$p^d, q, r^d, s, \neg t$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	

DPLL Example 2

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p^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate q)
p^d, q	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide r)
p^d, q, r^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate s)
p^d, q, r^d, s	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide t)
p^d, q, r^d, s, t^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate $\neg u$)
$p^d, q, r^d, s, t^d, \neg u$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Backtrack)
$p^d, q, r^d, s, \neg t$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide u)

DPLL Example 2

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p^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate q)
p^d, q	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide r)
p^d, q, r^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate s)
p^d, q, r^d, s	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide t)
p^d, q, r^d, s, t^d	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Propagate $\neg u$)
$p^d, q, r^d, s, t^d, \neg u$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Backtrack)
$p^d, q, r^d, s, \neg t$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	$\Rightarrow$ (Decide u)
$p^d, q, r^d, s, \neg t, u^d$	$\mid$	$\neg p \vee q,\ \neg r \vee s,\ \neg t \vee \neg u,\ u \vee \neg t \vee \neg q$	

Modern SAT Solvers

- DPLL + improvements
  - Backjumping
  - Dynamic variable ordering
  - Learning conflict clauses
  - Random restarts
  - …

Next Lecture

- First-order logic