

Lecture 3 First-Order Logic

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Last Time

- ▶ Propositional logic
 - ▶ Syntax and semantics
- ▶ Normal forms
 - ▶ NNF, DNF, CNF
- ▶ Tseitin transformation
- ▶ DPLL algorithm
 - ▶ Used in SAT solvers
- ▶ Homework assignment
 - ▶ Encode n-queens into SAT

First-Order Logic (FOL)

- ▶ Extends propositional logic with predicates, functions, and quantifiers
 - ▶ More expressive than PL
 - ▶ Suitable for reasoning about computation
- ▶ Examples
 - ▶ The length of one side of a triangle is less than the sum of the lengths of the other two sides

$$\forall x, y, z. \text{triangle}(x, y, z) \rightarrow \text{len}(x) < \text{len}(y) + \text{len}(z)$$
 - ▶ All elements of array A are 0

$$\forall i. 0 \leq i \wedge i < \text{size}(A) \rightarrow A[i] = 0$$

3

Syntax

variables x, y, z, \dots

constants a, b, c, \dots

functions f, g, h, \dots

terms variables, constants, or n-ary function applied to n terms as arguments

predicates p, q, r, \dots

atom \top, \perp , or n-ary predicate applied to n terms

literal atom or its negation

4

Syntax cont.

formula literal, application of a logical connective $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ to formulae, or application of a *quantifier* to a formula

▶ Quantifiers

- ▶ Existential: $\exists x. F[x]$
“there exists an x such that $F[x]$ ”
- ▶ Universal: $\forall x. F[x]$
“for all x , $F[x]$ ”

5

Example

$$\forall x. p(f(x), x) \rightarrow (\exists y. p(f(g(x, y)), g(x, y))) \wedge q(x, f(x))$$

6

Semantics

- ▶ An interpretation $I : (D_I, \alpha_I)$ is a pair
 - ▶ Domain D_I
 - ▶ Non-empty set of values or objects
 - ▶ Assignment α_I maps
 - ▶ each variable x into value $x_I \in D_I$
 - ▶ each n -ary function f into $f_I : D_I^n \rightarrow D_I$
 - ▶ each n -ary predicate p into $p_I : D_I^n \rightarrow \{\text{true}, \text{false}\}$
 - ▶ Boolean connectives evaluated as in propositional logic

7

Example

$$F : p(f(x,y),z) \rightarrow p(y,g(z,x))$$

Interpretation $I : (D_I, \alpha_I)$ with

$$D_I = \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\} \quad (\text{integers})$$

$$\alpha_I : \{ f \mapsto +, g \mapsto -, p \mapsto > \}$$

$$F_I : x + y > z \rightarrow y > z - x$$

$$\alpha_I : \{ x \mapsto 13, y \mapsto 42, z \mapsto 1 \}$$

$$F_I : 13 + 42 > 1 \rightarrow 42 > 1 - 13$$

Compute the truth value of F under I

1. $I \models x + y > z$ since $13 + 42 > 1$
2. $I \models y > z - x$ since $42 > 1 - 13$
3. $I \models F$ follows from 1, 2, and \rightarrow

F is true under I

8

Semantics of Quantifiers

- ▶ x -variant of interpretation $I : (D_I, \alpha_I)$ is an interpretation $J : (D_J, \alpha_J)$ such that
 - ▶ $D_I = D_J$
 - ▶ $\alpha_I[y] = \alpha_J[y]$ for all symbols y , except possibly x I and J agree on everything except maybe the value of x

- ▶ Denote $J : I \triangleleft \{x \mapsto v\}$ the x -variant of I in which $\alpha_J[x] = v$ for some $v \in D_I$. Then
 - ▶ $I \models \forall x.F$ iff for all $v \in D_I$, $I \triangleleft \{x \mapsto v\} \models F$
 - ▶ $I \models \exists x.F$ iff there exists $v \in D_I$ such that $I \triangleleft \{x \mapsto v\} \models F$

9

Example

- ▶ For $D_I = \mathbb{Q}$ (set of rational numbers), consider

$$F : \forall x. \exists y. 2 * y = x$$
- ▶ Compute the value of F_I :

Let

$$J_1 : I \triangleleft \{x \mapsto v\}$$
 be x -variant of I

$$J_2 : J_1 \triangleleft \{y \mapsto v/2\}$$
 be y -variant of J_1
 for $v \in \mathbb{Q}$.

Then

 1. $J_2 \models 2 * y = x$ since $2 * v/2 = v$
 2. $J_1 \models \exists y. 2 * y = x$
 3. $I \models \forall x. \exists y. 2 * y = x$ since $v \in \mathbb{Q}$ is arbitrary

10

Satisfiability and Validity

- ▶ F is **satisfiable** iff there exists I such that $I \models F$
 - ▶ F is **valid** iff for all I , $I \models F$
- F is valid iff $\neg F$ is unsatisfiable
- ▶ **FOL is undecidable**
 - ▶ There does not exist an algorithm for deciding if a FOL formula F is valid/unsat
 - ▶ I.e., that always halts and returns “yes” if F is valid/unsat or “no” if F is invalid/sat.
 - ▶ **FOL is semi-decidable**
 - ▶ There is a procedure that always halts and returns “yes” if F is valid, but may not halt if F is invalid.

11

Semantic Argument Method

- ▶ For proving validity of F in FOL
- ▶ Assume F is not valid and I is a falsifying interpretation: $I \not\models F$
- ▶ Exhaustively apply proof rules
 - ▶ If no contradiction reached and no more rules are applicable
 - ▶ F is invalid
 - ▶ If in every branch of proof a contradiction reached
 - ▶ F is valid

12

Proof Rule

- ▶ Consists of:
 - ▶ Premises (one or more)
 - ▶ Deductions (one or more)
- ▶ Application
 - ▶ Match premises to existing facts and form deductions
 - ▶ Branch (fork) when needed
- ▶ Example – proof rules for \wedge

$$\frac{I \models F \wedge G}{\begin{array}{l} I \models F \\ I \models G \end{array}} \quad \frac{I \not\models F \wedge G}{\begin{array}{l} I \not\models F \quad I \not\models G \end{array}}$$

13

Proof Rules for Propositional Part

$$\frac{I \models \neg F}{I \not\models F} \quad \frac{I \not\models \neg F}{I \models F} \quad \frac{I \models F \wedge G}{\begin{array}{l} I \models F \\ I \models G \end{array}} \quad \frac{I \not\models F \wedge G}{\begin{array}{l} I \not\models F \quad I \not\models G \end{array}}$$

$$\frac{I \models F \vee G}{\begin{array}{l} I \models F \quad I \models G \end{array}} \quad \frac{I \not\models F \vee G}{\begin{array}{l} I \not\models F \\ I \not\models G \end{array}} \quad \frac{I \models F \rightarrow G}{\begin{array}{l} I \not\models F \quad I \models G \end{array}} \quad \frac{I \not\models F \rightarrow G}{\begin{array}{l} I \models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \leftrightarrow G}{\begin{array}{l} I \models F \wedge G \quad I \not\models F \vee G \end{array}} \quad \frac{I \not\models F \leftrightarrow G}{\begin{array}{l} I \models F \wedge \neg G \quad I \models \neg F \wedge G \end{array}}$$

$$\frac{\begin{array}{l} I \models F \\ I \not\models F \end{array}}{I \models \perp}$$

14

Proof Rules for Quantifiers

$$\frac{I \models \forall x.F}{I \triangleleft \{x \mapsto v\} \models F} \text{ for any } v \in D_I$$

$$\frac{I \not\models \forall x.F}{I \triangleleft \{x \mapsto v\} \not\models F} \text{ for a fresh } v \in D_I$$

$$\frac{I \models \exists x.F}{I \triangleleft \{x \mapsto v\} \models F} \text{ for a fresh } v \in D_I$$

$$\frac{I \not\models \exists x.F}{I \triangleleft \{x \mapsto v\} \not\models F} \text{ for any } v \in D_I$$

any – introduced
earlier in the proof
fresh – has not been
previously used in
the proof

15

Example 1

$$F: p(a) \rightarrow \exists x. p(x)$$

16

Example 2

$$F: (\forall x. p(x)) \leftrightarrow (\neg \exists x. \neg p(x))$$

17

Next Lecture

- ▶ Issues with FOL
 - ▶ Validity in FOL is undecidable
 - ▶ Too general
- ▶ First-order logic theories
 - ▶ Often decidable fragments of FOL suitable for reasoning about particular domain
 - ▶ Equality
 - ▶ Arithmetic
 - ▶ Arrays

18